The Partial Simplicial Category and Algebras for Monads

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Abstract

We construct explicitly the weights on the simplicial category so that the colimits and limits of 2-functors with those weights provide the Kleisli objects and the Eilenberg-Moore objects, respectively, in any 2-category.

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1 Introduction

It is well know that monads in 2-categories correspond to 2-functors from the simplicial category Δ. It is also well known that the Kleisli and the Eilenberg-Moore objects can be build as weighted (co)limits, c.f. [St2]. In this paper we construct explicitly the weights W_r and W_l on Δ so that the W_r-weighted colimits provide the Kleisli objects and the W_l-weighted limits provide the Eilenberg-Moore objects in any 2-category $\mathcal{D}$.

2 Partial simplicial category Π

Let Δ be the usual (algebraists) simplicial category. The objects of Δ are finite linear orders denoted by $n = \langle n, \leq \rangle = \langle \{0, \ldots, n-1\}, \leq \rangle$, for $n \in \omega$. The morphisms of Δ are monotone functions. For $n \geq 1$ and $0 \leq i < n$, the morphism

$$\sigma_i^n : n+1 \longrightarrow n$$

is an epi that takes value i twice. For $n \geq 0$ and $0 \leq i \leq n$, the morphism

$$\delta_i^n : n \longrightarrow n+1$$

is a mono that misses the value i. We usually omit the upper index when it can be read from the context. These morphisms satisfy the following simplicial identities. For $i \leq j$

$$\delta_i\delta_j = \delta_{j+1}\delta_i \qquad\qquad \sigma_j\sigma_i = \sigma_i\sigma_{j+1}$$

and

$$\sigma_j\delta_i = \begin{cases} \delta_i\sigma_{j-1} & \text{if } i < j \\ 1 & \text{if } i = j, j+1 \\ \delta_{i-1}\sigma_j & \text{if } i > j+1 \end{cases}$$

It is well known, c.f [CWM], that the morphisms σ_i and δ_i generates Δ subject to the above relations.

The partial simplicial category Π has the same objects as Δ but the morphisms in Π are partial monotone functions. Clearly, the morphism σ_i and δ_i of Δ are morphism in Π as well, and they satisfy the same simplicial identities. For $n \geq 0$ and $0 \leq i \leq n$, the morphism

$$\tau_i^n : n+1 \longrightarrow n$$

is an epi not defined on i. For $i \leq j$ we have

$$\tau_j \tau_i = \tau_i \tau_{j+1}$$

Moreover, the morphisms σ_i, δ_i, and τ_i satisfy the following identities in Π that, together with the above identities, will be called *partial simplicial identities.*

$$\tau_j \sigma_i \ = \ \begin{cases} \sigma_i \tau_{j+1} & \text{if } i < j \\ \tau_j \tau_{j+1} & \text{if } i = j \\ \sigma_{i+1} \tau_j & \text{if } i > j \end{cases} \qquad \tau_j \delta_i \ = \ \begin{cases} \delta_i \tau_{j+1} & \text{if } i < j \\ 1 & \text{if } i = j \\ \delta_{i-1} \tau_j & \text{if } i > j \end{cases}$$

We have

Lemma 2.1. *Every morphism $f : n \to m$ in Π can be expressed in a canonical form as*

$$f = \delta_{i_r} \ldots \delta_{i_1} \sigma_{j_1} \ldots \sigma_{j_s} \tau_{k_1} \ldots \tau_{k_t} \tag{1}$$

with $i_1 < \ldots < i_r$, $j_1 < \ldots < j_s$, and $k_1 < \ldots < k_t$, $m - n = r - s - t$.

Proof. This can be easily seen directly or using the partial simplicial identities. □

Theorem 2.2. *The category Π is generated by the morphisms σ_i, δ_i, and τ_i subject to the partial simplicial identities.*

Proof. The partial simplicial identities hold in Π. Moreover, every morphism in Π can be written in a canonical form. Finally, two different canonical forms represent two different morphisms in Π. □

Remark. The category Π is a strict monoidal category with the monoidal structure defined by the coproduct. Moreover, the inclusion functor $\Delta \to \Pi$ is a strict morphism of strict monoidal categories.

3 The categories Π_l and Π_r and the multiplication functors

As the category Δ is a strict monoidal category it can be considered as 2-category with one 0-cell $*$, and then the tensor becomes the composition of 1-cells. We denote this 2-category by $\mathbf{\Delta}$.

The *left partial simplicial category* Π_l is a subcategory of Π with the same objects as Π. A morphism $f : n \to m$ from Π is in Π_l iff for any $i \leq j \in n$, if $f(j)$ is defined so is $f(i)$. In other words the morphisms of Π are generated by the morphisms in Δ and the morphism $\tau_n^n : n+1 \to n$ for $n \in \omega$. Note that $\tau_n^n = id_n + \tau_0^0$. In Π_l Lemma 2.1 holds with an additional condition that $k_{i+1} = k_i + 1$ and $k_t = n - 1$.

We have a *left multiplication* 2-functor

$$W_l : \mathbf{\Delta} \to 2Cat$$

such that

$$W_l(*) = \Pi_l, \quad W_l(n) = n + (-), \quad W_l(f) = f + (-)$$

for $f : n \to m \in \Delta$. Thus W_l can be seen as an action od Δ on Π_l by tensoring on the left. Clearly, Π_l is closed with respect to such operations.

Dually, we have the *right partial simplicial category* Π_r, a subcategory of Π with the same objects as Π. A morphism $f : n \to m$ from Π is in Π_r iff for any $i \leq j \in n$, if $f(i)$ is defined so is $f(j)$. In other words the morphisms of Π are generated by the morphisms in Δ and the morphism $\tau_0^n : n+1 \to n$, for $n \in \omega$. Note that $\tau_0^n = \tau_0^0 + id_n$. In Π_l Lemma 2.1 holds with a additional condition that $k_{i+1} = k_i + 1$ and $k_1 = 0$.

We have a *right multiplication* 2-functor

$$W_r : \mathbf{\Delta} \to 2Cat$$

such that

$$W_r(*) = \Pi_r, \quad W_r(n) = (-) + n, \quad W_r(f) = (-) + f$$

for $f : n \to m \in \Delta$. Thus W_r can be seen as an action od Δ on Π_r by tensoring on the right. Clearly, Π_r is closed with respect to such operations.

4 Monads as 2-functors

The 2-functors $\mathbf{T} : \mathbf{\Delta} \to \mathcal{D}$ correspond bijectively to monads in the 2-category $\mathcal{D}$.

Suppose $(\mathcal{C}, T, \eta, \mu)$ is a monad in $\mathcal{D}$ on $\mathcal{C}$. We define a 2-functor $\mathbf{T}$ as follows:

$$\mathbf{T}(*) = \mathcal{C}, \quad \mathbf{T}(0) = 1_{\mathcal{C}}, \quad \mathbf{T}(n) = T^n,$$

$$\mathbf{T}(\delta_0^0) = \eta, \quad \mathbf{T}(\delta_i^n) = T^{n-i}\eta_{T^i}, \quad \mathbf{T}(\sigma_0^1) = \mu, \quad \mathbf{T}(\sigma_i^n) = T^{n-i-1}\mu_{T^i},$$

The equations

$$\mathbf{T}(\sigma_i) \circ \mathbf{T}(\sigma_i) = \mathbf{T}(\sigma_i) \circ \mathbf{T}(\sigma_{i+1})$$

hold, as the consequence of the associativity of the multiplication $\mu \circ T\mu = \mu \circ \mu_T$. The equations

$$\mathbf{T}(\sigma_i) \circ \mathbf{T}(\delta_i) = 1 = \mathbf{T}(\sigma_{i+1}) \circ \mathbf{T}(\delta_i)$$

hold, as the consequence of the unit axiom $\mu \circ T\eta = 1 = \mu \circ \eta_T$. The remaining simplicial equations hold as a consequence of the Middle Exchange Law (MEL).

On the other hand, having a 2-functor $\mathbf{T} : \mathbf{\Delta} \to \mathcal{D}$, we get a monad $(\mathbf{T}(*), \mathbf{T}(1), \mathbf{T}(\delta_0^0), \mathbf{T}(\sigma_0^1))$.

Let $2\mathbf{Cat}$ be the 3-category of 2-categories. As $2\mathbf{Cat}(\mathbf{\Delta}, \mathcal{D})$ is the 2-category of monads in $\mathcal{D}$ with strict morphisms, we can think of $\mathbf{\Delta}$ as a 2-category representing monads with strict morphisms in 2-categories.

5 The 2-functor $Subeq_T$ and the Eilenberg-Moore objects

For a given monad $(\mathcal{C}, T, \eta, \mu)$ in a 2-category $\mathcal{D}$ we define a 2-functor

$$Subeq_T : \mathcal{D}^{op} \longrightarrow \mathbf{Cat}$$

as follows. For a given 0-cell X in $\mathcal{D}$, the category $Subeq_T(X)$ has as objects pairs (U, ξ) such that $U : X \to \mathcal{C}$ is a 1-cell in $\mathcal{D}$, $\xi : TU \to U$ is a 2-cell in $\mathcal{D}$ such that in the diagram

$$\begin{tikzcd} T^2U \arrow[r, shift left, "T(\xi)"] \arrow[r, shift right, "\mu_U"'] & TU \arrow[r, shift left, "\xi"] & U \arrow[l, shift left, "\eta_U"] \end{tikzcd}$$

we have

$$\xi \circ \eta_U = 1_U, \qquad \xi \circ T(\xi) = \xi \circ \mu_U.$$

In such case, we say that (U,ξ) *subequalizes the monad* T. A morphism $\tau : (U,\xi) \to (U',\xi')$ is a 2-cell $\tau : U \to U'$ such that the square

$$\begin{tikzcd} TU \arrow[r, "T(\tau)"] \arrow[d, "\xi"'] & TU' \arrow[d, "\xi'"] \\ U \arrow[r, "\tau"'] & U' \end{tikzcd}$$

commutes. The 2-functor $Subeq_T$ is define on 1- and 2-cells in the obvious way, by composition.

Recall, from [St1], see [Z] for the notation, that *the* 2*-category* $\mathcal{D}$ *admits Eilenberg-Moore objects* if the embedding ι

$$\begin{tikzcd} \mathcal{D} \arrow[r, shift left, "\iota"] & \mathbf{Mnd}(\mathcal{D}) \arrow[l, shift left, "EM"] \end{tikzcd}$$

has a right 2-adjoint $\iota \dashv EM$. $\mathbf{Mnd}(\mathcal{D})$ is the 2-category of monads in $\mathcal{D}$ with lax morphisms of monads and transformations of lax morphisms. We have a 2-functor

$$\mathbf{Mnd}(\iota(-),T) : \mathcal{D}^{op} \longrightarrow \mathbf{Cat} \tag{2}$$

sending 0-cell X in $\mathcal{D}$ to the category $\mathbf{Mnd}(\iota(X),T)$ of lax morphisms from the identity monad on X to the monad T and transformations between such morphisms.

The following definition is a 'monad by monad' version of the previous one. We say that *the monad* T *admits Eilenberg-Moore object* iff the 2-functor $\mathbf{Mnd}(\iota(-),T)$ is representable.

A simple verification shows

Lemma 5.1. *The* 2*-functors* $Subeq_T$ *and* $\mathbf{Mnd}(\iota(-),T)$ *are naturally isomorphic.*

6 The 2-functor $Cone_{W_l}(\mathbf{T})$

Let $\mathbf{T} : \Delta \to \mathcal{D}$ be a 2-functor and T be the monad corresponding to $\mathbf{T}$. We define a 2-functor

$$Cone_{W_l}(\mathbf{T}) : \mathcal{D}^{op} \longrightarrow \mathbf{Cat}$$

of W_l-cones over $\mathbf{T}$. We will show that the 2-functor $Cone_{W_l}(\mathbf{T})$ is isomorphic to the 2-functor $Subeq_T$.

Fix a 0-cell X in $\mathcal{D}$. An object of $Cone_{W_l}(\mathbf{T})(X)$ is a 2-natural transformation

$$\lambda : W_l \longrightarrow \mathcal{D}(X,\mathbf{T}(*))$$

with only one component the functor λ_* also denoted λ. The 2-naturality means that the square

$$\begin{tikzcd}[column sep=huge] \Pi_l \arrow[r, "\lambda"] \arrow[d, bend right, "m+(-)"'] \arrow[d, bend left, "n+(-)"] \arrow[d, phantom, "\overset{f+(-)}{\Longrightarrow}"] & \mathcal{D}(X,\mathbf{T}(*)) \arrow[d, bend right, "\mathcal{D}(X{,}\mathbf{T}(m))"'] \arrow[d, bend left, "\mathcal{D}(X{,}\mathbf{T}(n))"] \arrow[d, phantom, "\overset{\mathcal{D}(X{,}\mathbf{T}(f))}{\Longrightarrow}"] \\ \Pi_l \arrow[r, "\lambda"'] & \mathcal{D}(X,\mathbf{T}(*)) \end{tikzcd}$$

commutes, for any $f : m \to n$ in Δ. Put $T = \mathbf{T}(1)$ and $\lambda(0) = U : X \to \mathbf{T}(*)$. We have with f as before

$$\lambda(m) = \lambda(m + 0) = \mathbf{T}(m) \circ \lambda(0) = T^m U,$$

$$\lambda(f) = \lambda(f + 0) = \mathbf{T}(f) \circ \lambda(0) = \mathbf{T}(f) \circ U$$

Moreover, putting $\lambda(\tau_0^0) = \xi : TU \to U$, we have

$$\lambda(\tau_n^n) = \lambda(id_n + \tau_0^0) = \mathbf{T}(id_n) \circ_0 \mathbf{T}(\tau_0^0) = \mathbf{T}(id_n) \circ_0 \xi =$$

$$= \mathbf{T}(id_1) \circ_0 \ldots \circ_0 \mathbf{T}(id_1) \circ_0 \xi = id_T \circ_0 \ldots \circ_0 id_T \circ_0 \xi = T^n(\xi)$$

Thus λ is uniquely determined by U and ξ. The equations

$$\tau_0^0 \circ \delta_0^1 = 1, \qquad \tau_0^0 \circ \tau_1^1 = \tau_0^0 \circ \sigma_0^1$$

implies that (U, ξ) subequalizes the monad T. On the other hand, if (U, ξ) subequalizes T then we can define a 2-natural transformation $\lambda : W_l \to \mathcal{D}(X, \mathbf{T})$, as follows. The functor

$$\lambda = \lambda_* : \Pi_l \longrightarrow \mathcal{D}(X, \Pi(*))$$

is defined so that

$$\lambda(0) = U, \;\; \lambda(\tau_0^0) = \xi$$

and for 2-naturality of λ we have

$$\lambda(n) = T^n U, \;\; \lambda(f) = \mathbf{T}(f)_U, \;\; \lambda(\tau_n^n) = T^n(\xi)$$

Then it is easy to verify that λ respect all the equations in Π_l.

A morphism in $Cone_{W_l}(\mathbf{T})(X)$ between two 2-natural transformations is a modification $\nu : \lambda \to \lambda'$ with one component ν_*, denoted also ν, that is a natural transformation so that, for any $n \in \omega$, the square

$$\begin{tikzcd}
\Pi_l \arrow[r, bend left, "\lambda"] \arrow[r, bend right, "\lambda'"'] \arrow[r, phantom, "\nu\Downarrow"] \arrow[d, "n+(-)"'] & \mathcal{D}(X, \mathbf{T}(*)) \arrow[d, "\mathcal{D}(X, \mathbf{T}(n))"] \\
\Pi_l \arrow[r, bend left, "\lambda"] \arrow[r, bend right, "\lambda'"'] \arrow[r, phantom, "\nu\Downarrow"] & \mathcal{D}(X, \mathbf{T}(*))
\end{tikzcd}$$

commutes. As we have

$$\nu_n = \nu_{(n+0)} = T^n(\nu_0)$$

the modification ν is uniquely determined by $\nu_0 : U \to U' = \lambda'(0)$. The square

$$\begin{tikzcd}
\lambda(1) \arrow[r, "\nu_1"] \arrow[d, "\lambda(\tau_0^0)"'] & \lambda'(1) \arrow[d, "\lambda'(\tau_0^0)"] \\
\lambda(0) \arrow[r, "\nu_0"'] & \lambda'(0)
\end{tikzcd}$$

commutes as it is the naturality of $\nu_* : \lambda_* \to \lambda'_*$ on τ_0^0.

On the other hand, any 2-cell $\nu_0 : U \to U'$ in $\mathcal{D}$ such that

$$\begin{tikzcd} TU \arrow[r, "T(\nu_0)"] \arrow[d, "\xi"'] & TU' \arrow[d, "\xi'"] \\ U \arrow[r, "\nu_0"'] & U' \end{tikzcd}$$

extends to a natural transformation from λ_* to λ'_*, i.e. a modification ν from λ to λ'.

The 2-functor $Cone_{W_l}(\mathbf{T})$ is defined on 1- and 2-cells in the obvious way.

Constructing this functor we have in fact proved

Lemma 6.1. *The* 2*-functors* $Subeq_T$ *and* $Cone_{W_l}(\mathbf{T})$ *are naturally isomorphic.*

7 The Eilenberg-Moore objects

Theorem 7.1. *Let* $(\mathcal{C}, T, \eta, \mu)$ *be a monad in a* 2*-category* $\mathcal{D}$ *and* $\mathbf{T} : \mathbf{\Delta} \to \mathcal{D}$ *the corresponding* 2*-functor. Then* T *admits Eilenberg-Moore object iff* $\mathbf{T}$ *has a* W_l*-weighted limit. If it is the case then the Eilenberg-Moore object for* T *and the* W_l*-weighted limit of* $\mathbf{T}$ *are isomorphic.*

Proof. By Lemmas 5.1 and 6.1, the 2-functors $\mathbf{Mnd}(\iota(-), T)$ and $Cone_{W_l}(\mathbf{T})$ are naturally isomorphic. So if one is of representable so is the other and the representing objects are isomorphic. The representation of the first give rise to the Eilenberg-Moore object for T, and the representation of the second give rise to the W_l-weighted limit of $\mathbf{T}$. $\square$

From the above theorem we get immediately

Corollary 7.2. *Any* 2*-category* $\mathcal{D}$ *admits Eilenberg-Moore object iff it has all* W_l*-weighted limit of* 2*-functors from* Δ.

8 The Kleisli objects

Clearly, all the above considerations can be dualised. In this case we get results relating Kleisli objects and the W_r-weighted colimits of 2-functors from Δ.

We note for the record

Theorem 8.1. *Let* $(\mathcal{C}, T, \eta, \mu)$ *be a monad in a* 2*-category* $\mathcal{D}$ *and* $\mathbf{T} : \Delta \to \mathcal{D}$ *the corresponding* 2*-functor. Then* T *admits Kleisli object iff* $\mathbf{T}$ *has a* W_r*-weighted colimit. If it is the case then the Kleisli object for* T *and the* W_r*-weighted colimit of* $\mathbf{T}$ *are isomorphic.*

Corollary 8.2. *Any* 2*-category* $\mathcal{D}$ *admits Kleisli object iff it has all* W_r*-weighted colimit of* 2*-functors from* Δ.

9 Appendix: Weighted limits in 2-categories

We recall the definition of weighted limits in 2-categories in detail.

The 2-functor $\mathcal{D}(X, \mathbf{T})$

For two 2-functors between 2-categories as shown[1]

[1] There are some foundational problems that one should address. For example, it is desirable that the 2-category $\mathcal{I}$ be small. But we will be ignoring this issues believing that the reader can fix all these problem on its own, the way she or he likes most.

$$\mathcal{I} \xrightarrow{W} Cat \qquad \mathcal{I} \xrightarrow{\mathbf{T}} \mathcal{D}$$

we are going to describe the W-weighted limit of $\mathbf{T}$.

For any 0-cell X in $\mathcal{D}$ we can form a 2-functor

$$\begin{tikzcd}
\mathcal{I} \arrow[rrr, "\mathcal{D}(X,\mathbf{T})"] & & & \mathbf{Cat} \\
i \arrow[d, bend right, "f"'] \arrow[d, bend left, "g"] \arrow[d, phantom, "\overset{\alpha}{\Longrightarrow}"] & \longmapsto & & \mathcal{D}(X,\mathbf{T}(i)) \arrow[d, bend right, "\mathcal{D}(X,\mathbf{T}(f))"'] \arrow[d, bend left, "\mathcal{D}(X,\mathbf{T}(f))"] \arrow[d, phantom, "\overset{\mathcal{D}(X,\mathbf{T}(\alpha))}{\Longrightarrow}"] \\
j & & & \mathcal{D}(X,\mathbf{T}(j))
\end{tikzcd}$$

of 'homming into' $\mathbf{T}$.

The category $\mathcal{D}(X,\mathbf{T}(i))$ consists of 1- and 2-cells in $\mathcal{D}$ from X to $\mathbf{T}(i)$.

The functor

$$\mathcal{D}(X,\mathbf{T}(i)) \xrightarrow{\mathcal{D}(X,\mathbf{T}(f))} \mathcal{D}(X,\mathbf{T}(j))$$

is a whiskering along the 2-cell $\mathbf{T}(f)$:

$$\begin{tikzcd}
X \arrow[r, bend left, "r"] \arrow[r, bend right, "s"'] \arrow[r, phantom, "\gamma\Downarrow"] & \mathbf{T}(i) & \longmapsto & X \arrow[r, bend left, "\mathbf{T}(f)\circ r"] \arrow[r, bend right, "\mathbf{T}(f)\circ s"'] \arrow[r, phantom, "\mathbf{T}(f)(\gamma)\Downarrow"] & \mathbf{T}(j)
\end{tikzcd}$$

The component of the natural transformation

$$\mathcal{D}(X,\mathbf{T}(f)) \xrightarrow{\mathcal{D}(X,\mathbf{T}(\alpha))} \mathcal{D}(X,\mathbf{T}(g))$$

at $r : X \to \mathbf{T}(i)$ is

$$\mathbf{T}(f)\circ r \xrightarrow{\mathbf{T}(\alpha)_r} \mathbf{T}(g)\circ r$$

The naturality of $\mathcal{D}(X,\mathbf{T}(f))$

$$\begin{tikzcd}
\mathbf{T}(f)\circ r \arrow[r, "\mathbf{T}(\alpha)_r"] \arrow[d, "\mathbf{T}(f)(\gamma)"'] & \mathbf{T}(g)\circ r \arrow[d, "\mathbf{T}(g)(\gamma)"] \\
\mathbf{T}(f)\circ s \arrow[r, "\mathbf{T}(\alpha)_s"'] & \mathbf{T}(f)\circ r
\end{tikzcd}$$

follows from MEL, where

$$\begin{tikzcd}
X \arrow[r, bend left, "r"] \arrow[r, bend right, "s"'] \arrow[r, phantom, "\gamma\Downarrow"] & \mathbf{T}(i) \arrow[r, bend left, "\mathbf{T}(f)"] \arrow[r, bend right, "\mathbf{T}(g)"'] \arrow[r, phantom, "\mathbf{T}(\alpha)\Downarrow"] & \mathbf{T}(j)
\end{tikzcd}$$

This ends the definition of the 2-functor $\mathcal{D}(X,\mathbf{T})$.

The 2-functor of weighted cones

Using the above 2-functor(s) we can form the 2-functor $Cone_W(\mathbf{T})$ of W-cones over $\mathbf{T}$.

$$\begin{tikzcd} \mathcal{D}^{op} \arrow[rrr, "Cone_W(\mathbf{T})"] & & & \mathbf{Cat} \end{tikzcd}$$

$$\begin{tikzcd} X \arrow[d, "F"'] \arrow[d, bend left=0, phantom, ""] & & & Cone_W(\mathbf{T})(X) \\ Y & \longmapsto & & Cone_W(\mathbf{T})(Y) \arrow[u, "Cone_W(\mathbf{T})(F)"] \end{tikzcd} \quad \begin{tikzcd} X \arrow[d, "G"] \\ Y \end{tikzcd} \quad \beta: F \Rightarrow G, \quad Cone_W(\mathbf{T})(\beta): Cone_W(\mathbf{T})(F) \Rightarrow Cone_W(\mathbf{T})(G)$$

Fix X in $\mathcal{D}$. The category $Cone_W(\mathbf{T})(X)$ consists of 2-natural transformations between 2-functors W and $\mathcal{D}(X, \mathbf{T})$ and modifications between them.

The objects in the category $Cone_W(\mathbf{T})(X)$ are 2-natural transformations

$$\begin{tikzcd} W \arrow[rr, "\lambda"] & & \mathcal{D}(X, \mathbf{T}) \end{tikzcd}$$

$$\mathcal{I} \ni \begin{tikzcd} i \arrow[d, bend right, "f"'] \arrow[d, bend left, "g"] \arrow[d, phantom, "\overset{\alpha}{\Rightarrow}"] \\ j \end{tikzcd} \qquad \begin{tikzcd}[column sep=huge] W_i \arrow[r, "\lambda_i"] \arrow[d, bend right, "W_f"'] \arrow[d, bend left, "W_g"] \arrow[d, phantom, "\overset{W_\alpha}{\Longrightarrow}"] & \mathcal{D}(X, \mathbf{T}(i)) \arrow[d, bend right, "\mathcal{D}(X,\mathbf{T}(f))"'] \arrow[d, bend left, "\mathcal{D}(X,\mathbf{T}(g))"] \arrow[d, phantom, "\overset{\mathcal{D}(X,\mathbf{T}(\alpha))}{\Longrightarrow}"] \\ W_j \arrow[r, "\lambda_j"'] & \mathcal{D}(X, \mathbf{T}(j)) \end{tikzcd}$$

so that

$$\mathcal{D}(X, \mathbf{T}(\alpha)) \circ \lambda_i = \lambda_j \circ W_\alpha$$

The morphisms in the category $Cone_W(\mathbf{T})(X)$ are modifications $\nu : \lambda \to \lambda'$ or

$$\begin{tikzcd} W \arrow[r, bend left, "\lambda"] \arrow[r, bend right, "\lambda'"'] \arrow[r, phantom, "\nu \Downarrow"] & \mathcal{D}(X, \mathbf{T}) \end{tikzcd}$$

such that, for $f : i \to j$ in $\mathcal{I}$, the square

$$\begin{tikzcd}[column sep=huge] W_i \arrow[r, bend left, "\lambda_i"] \arrow[r, bend right, "\lambda'_i"'] \arrow[r, phantom, "\nu_i \Downarrow"] \arrow[d, "W_f"'] & \mathcal{D}(X, \mathbf{T}(i)) \arrow[d, "\mathcal{D}(X,\mathbf{T}(f))"] \\ W_j \arrow[r, bend left, "\lambda_j"] \arrow[r, bend right, "\lambda'_j"'] \arrow[r, phantom, "\nu_j \Downarrow"] & \mathcal{D}(X, \mathbf{T}(j)) \end{tikzcd}$$

commutes, in the sense that

$$\mathcal{D}(X, \mathbf{T}(f)) \circ \nu_i = \nu_j \circ W_f$$

This ends the definition of the category $Cone_W(\mathbf{T})(X)$.

The functor

$$\begin{tikzcd} Cone_W(\mathbf{T})(X) \arrow[rr, "Cone_W(\mathbf{T})(F)"] & & Cone_W(\mathbf{T})(Y) \end{tikzcd}$$

sends the 2-natural transformation λ to the 2-natural transformation

$$W \xrightarrow{\lambda} \mathcal{D}(Y,\mathbf{T}) \xrightarrow{\mathcal{D}(F,\mathbf{T})} \mathcal{D}(X,\mathbf{T})$$

such that, for i in $\mathcal{I}$,

$$W_i \xrightarrow{\lambda_i} \mathcal{D}(Y,\mathbf{T}(i)) \xrightarrow{\mathcal{D}(F,\mathbf{T}(i))} \mathcal{D}(X,\mathbf{T}(i))$$

is a functor such that, for $u : w \to w'$ in W_i, we have a diagram

$$\begin{tikzcd} X \arrow[r, "F"] & Y \arrow[r, bend left, "\lambda_i(w)"] \arrow[r, bend right, "\lambda_i(w')"'] \arrow[r, phantom, "\lambda_i(u)\Downarrow"] & \mathbf{T}(i) \end{tikzcd}$$

and the following equations

$$\mathcal{D}(F,\mathbf{T}(i)) \circ \lambda_i(w) = \lambda_i(w) \circ F$$

$$\mathcal{D}(F,\mathbf{T}(i)) \circ \lambda_i(u) = \lambda_i(u)_F$$

hold. Moreover, the functor $Cone_W(\mathbf{T})(F)$ sends the modification ν

$$\begin{tikzcd} W \arrow[r, bend left, "\lambda"] \arrow[r, bend right, "\lambda'"'] \arrow[r, phantom, "\nu\Downarrow"] & \mathcal{D}(Y,\mathbf{T}) \end{tikzcd}$$

to the modification

$$\begin{tikzcd} W \arrow[r, bend left, "Cone_W(\mathbf{T})(F)(\lambda)=\bar{\lambda}"] \arrow[r, bend right, "Cone_W(\mathbf{T})(F)(\lambda')=\bar{\lambda}'"'] \arrow[r, phantom, "Cone_W(\mathbf{T})(F)(\nu)=\bar{\nu}\Downarrow"] & \mathcal{D}(X,\mathbf{T}) \end{tikzcd}$$

such that, for i in $\mathcal{I}$,

$$\begin{tikzcd} W \arrow[r, bend left, "\bar{\lambda}_i"] \arrow[r, bend right, "\bar{\lambda}'_i"'] \arrow[r, phantom, "\bar{\nu}_i\Downarrow"] & \mathcal{D}(X,\mathbf{T}(i)) \end{tikzcd}$$

is a natural transformation such, that for w in W_i,

$$\bar{\lambda}_i(w) = \lambda_i(w) \circ F \xrightarrow{(\bar{\nu}_i)_w) = ((\nu_i)_w)_F} \lambda'_i(w) \circ F = \bar{\lambda}'_i(w)$$

is a morphism in $\mathcal{D}(X,\mathbf{T}(i))$.

The component, at the 2-natural transformation $\lambda : W \to \mathcal{D}(X,\mathbf{T})$, of the natural transformation

$$Cone_W(\mathbf{T})(F) \xrightarrow{Cone_W(\mathbf{T})(\beta)} Cone_W(\mathbf{T})(G)$$

is a modification $\mathcal{D}(\beta,\mathbf{T}) \circ \lambda$, i.e. the composition

$$\begin{tikzcd} W \arrow[r, "\lambda"] & \mathcal{D}(Y,\mathbf{T}) \arrow[r, bend left, "\mathcal{D}(F,\mathbf{T})"] \arrow[r, bend right, "\mathcal{D}(G,\mathbf{T})"'] \arrow[r, phantom, "\mathcal{D}(\beta,\mathbf{T})\Downarrow"] & \mathcal{D}(X,\mathcal{T}) \end{tikzcd}$$

so that, at i in $\mathcal{I}$, it is the natural transformation $\mathcal{D}(\beta,\mathbf{T}(i)) \circ \lambda_i$

$$\begin{tikzcd} W_i \arrow[r, "\lambda_i"] & \mathcal{D}(Y,\mathbf{T}(i)) \arrow[r, bend left, "\mathcal{D}(F,\mathbf{T}(i))"] \arrow[r, bend right, "\mathcal{D}(G,\mathbf{T}(i))"'] \arrow[r, phantom, "\mathcal{D}(\beta,\mathbf{T}(i))\Downarrow"] & \mathcal{D}(X,\mathbf{T}(i)) \end{tikzcd}$$

so that, for w in W_i, it is a morphism in $\mathcal{D}(X,\mathbf{T})$

$$\lambda_i(w) \circ F \xrightarrow{\lambda_i(w)(\beta)} \lambda_i(w) \circ G$$

from the diagram

$$\begin{tikzcd} X \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\beta\Downarrow"] & Y \arrow[r, "\lambda_i(w)"] & \mathbf{T}(i) \end{tikzcd}$$

The representation of the 2-functor $Cone_W(\mathbf{T})$

The representation of the functor $Cone_W(\mathbf{T})$ is the W-weighted limit of the 2-functor $\mathbf{T}$. Thus it is an object $Lim_W(\mathbf{T})$ together with a 2-natural isomorphism

$$\mathcal{D}(-, Lim_W(\mathbf{T})) \xrightarrow{\varrho} Cone_W(\mathbf{T})$$

The image of the identity on $Lim_W(\mathbf{T})$ is the limiting W-weighted cone

$$Lim_W(\mathbf{T}) \xrightarrow{\pi} \mathbf{T}$$

in $Cone_W(\mathbf{T})(Lim_W(\mathbf{T}))$. For any 0-cell X we have a correspondence via π

$$\begin{tikzcd}
X \arrow[rr, shift left=2, "L"] \arrow[rr, shift right=2, "L'"'] \arrow[rr, phantom, "n \Downarrow"] \arrow[d, shift right=2, "\lambda"'] \arrow[d, shift left=2, "\lambda'"] \arrow[d, phantom, "\overset{\nu}{\Rightarrow}"] & & Lim_W(\mathbf{T}) \arrow[dll, "\pi"] \\
\mathbf{T} & &
\end{tikzcd}$$

or in another form, we have an isomorphism of categories

$$\begin{array}{cc}
\begin{tikzcd} X \arrow[r, shift left=2, "L"] \arrow[r, shift right=2, "L'"'] \arrow[r, phantom, "n \Downarrow"] & Lim_W(\mathbf{T}) \end{tikzcd} & \text{in } \mathcal{D} \\
\hline
\begin{tikzcd} X \arrow[r, shift left=2, "\lambda"] \arrow[r, shift right=2, "\lambda'"'] \arrow[r, phantom, "\nu \Downarrow"] & \mathbf{T} \end{tikzcd} & \text{in } Cone_W(\mathbf{T})(X)
\end{array}$$

natural in X.

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